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On the Multifractal Analysis of Measures

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The multifractal formalism is shown to hold for a large class of measures.

KEY WORDS: Multifractals; Hausdorff dimension; Tricot dimension; large deviations.

1. INTRODUCTION: THE MULTIFRACTAL FORMALISM

Let μ be a Borel probability measure on $[0, 1]$. Suppose that, for every $q \in \mathbb{R}$, the following quantity exists:

$$\tau(q) = \lim_{b \to \infty} \frac{-1}{\log N_n} \log \sum_{0 \leqslant j < N_n}{}' \left[\mu\left(\left[\frac{j}{N_n}, \frac{j+1}{N_n} \right[\right) \right]^q$$

where N_n is an increasing sequence of integers and the prime means that the summation runs through those indices j such that

$$\mu\left(\left[\frac{j}{N_n}, \frac{j+1}{N_n} \right[\right) \neq 0$$

On the other hand, consider the set

$$E_\alpha = \left\{ t \in [0, 1[;\ \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \to \alpha \right\}$$

where $\alpha \in \mathbb{R}$ and $I_n(t)$ is the interval $[j/N_n, (j+1)/N_n[$ which contains t.

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Then it is asserted,[9–11] and proved in certain cases,[1,2,5,7,13] that

$$\dim E_\alpha = f(\alpha) := \inf_{q \in \mathbb{R}} (\alpha q - \tau(q))$$

where dim stands for a suitable notion of dimension. In the case where $\alpha = \tau'(q)$, then $\dim E_\alpha = \alpha q - \tau(q)$.

We are grateful to the referee for suggesting that we include a heuristic argument along the following lines.

For a pure fractal we would have $\mu(I_i) \simeq |I_i|^\alpha$ and $\sum |I_i|^\alpha = 1$, for some fixed α which gives the dimension. For a multifractal, we have the local formalism, $\mu(I_i) = |I_i|^{\alpha_i}$, which gives $\sum |I_i|^{\alpha_i q - \tau(q)} \equiv 1$.

Now fix arbitrary α and minimize the function $q \to \alpha q - \tau(q)$. Suppose the minimum occurs at q_0. We have

$$\sum_{\alpha_i = \alpha} |I_i|^{f(\alpha)} + \sum_{\alpha_i \neq \alpha} |I_i|^{\alpha_i q_0 - \tau(q_0)} \equiv 1$$

In the case where the contribution from the second term is relatively negligible (this can be investigated technically using large deviations) we have the formalism $\sum_{\alpha_i = \alpha} |I_i|^{f(\alpha)} = 1$, which demonstrates that the dimension of the set E_α is calculated by the formula $f(\alpha)$.

Our aim is twofold: first, to examine what can be said in general, without making restrictive assumptions on μ; second, to define a setting in which the multifractal formalism works. The remainder of this article is organized as follows.

In Section 2 we establish some large-deviation results. In Section 3, we show that, in general, instead of the equality, an inequality holds. In Section 4, we define a class of measures for which the formalism is valid. These measures have already been consisdered by one of the authors[19] and contain as a particular case the multinomial measures, described in Section 5, and other measures occurring in certain dynamical systems, such as "cookie cutters." In Section 5 we give some examples and applications.

Our methods do not appear to extend naturally to discuss further interpretations of $f(\alpha)$ as in refs. 14–18. The use of partition functions for computing Hausdorff dimensions also appeared in refs. 12 and 20.

2. CHERNOFF-TYPE RESULTS

Let $(\Omega_n, \mathscr{A}_n, \mu_n)$ be a sequence of probability spaces, $\{\lambda_n\}_{n \geq 1}$ a sequence of positive numbers, and $\{u_n\}$ and $\{v_n\}$ two sequences of random variables with values in $[0, 1]$, μ_n and v_n being $\mathscr{A}_n$-measurable.

Let X_n be the set $X_n = \{u_n v_n \neq 0\}$. For real numbers x and y, set

$$C_n(x, y) = \lambda_n^{-1} \log \int_{X_n} u_n^x v_n^{-y} \, d\mu_n$$

and

$$C(x, y) = \limsup_{n \to \infty} C_n(x, y)$$

It is well known that C_n and C are convex functions. Consider the set $\Omega = \{(x, y) \in \mathbb{R}^2;\ C(x, y) < 0\}$ and its interior $\mathring{\Omega}$. Since $C(x, y)$ is nonincreasing as a function of x and nondecreasing as a function of y, the set Ω, if it contains a point (a, b), also contains the whole quadrant $\{(a+x, b-y);\ x \geqslant 0 \text{ and } y \geqslant 0\}$. It results that there exists a concave and nondecreasing function φ from $\mathbb{R}$ to $\bar{\mathbb{R}}$ such that

$$\mathring{\Omega} = \{(x, y) \in \mathbb{R}^2;\ y < \varphi(x-0)\}$$

If $\Omega = \varnothing$, then φ is identically equal to $-\infty$; if $\Omega = \mathbb{R}^2$, then φ is identically equal to $+\infty$; if $\mathring{\Omega}$ is the half-plane $\{(x, y);\ x > x_0\}$, then $\varphi(x) = -\infty$ for $x < x_0$ and $\varphi(x) = +\infty$ for $x > x_0$.

From now on, we assume that φ is finite on an open interval I containing 0.

For any $\gamma \in \mathbb{R}$, we consider the following Legendre transform of φ:

$$f_\gamma(\alpha) = \inf_{x \in \mathbb{R}} [\alpha(x-\gamma) - \varphi(x)] = f_0(\alpha) - \alpha\gamma$$

For $\gamma \in I$, the maximum value of $f_\gamma(\alpha)$ is $-\varphi(\gamma)$ and is assumed for $\alpha \in [\varphi'(\gamma+0), \varphi'(\gamma-0)]$. The function f_γ is nondecreasing on the interval $]-\infty, \varphi'(\gamma+0)]$, and nonincreasing on the interval $[\varphi'(\gamma-0), +\infty[$. The two following remarks will be useful:

1. If $\alpha \leqslant \varphi'(\gamma-0)$ and $\delta > f_\gamma(\alpha)$, then there exists $t > 0$ such that $C(\gamma+t, -\delta+\alpha t) < 0$,
2. If $\alpha \geqslant \varphi'(\gamma+0)$ and $\delta > f_\gamma(\alpha)$, then there exists $t > 0$ such that $C(\gamma-t, -\delta-\alpha t) < 0$.

The following results can be thought as being a generalized form of Chernoff inequality.[6]

Proposition 1. For any $\gamma \in I$, we have the following facts:

1. If $\alpha \leqslant \varphi'(\gamma-0)$ and $\delta > f_\gamma(\alpha)$, then

$$\limsup \frac{1}{\lambda_n} \log \int_{X_n \cap \{u_n \geqslant v_n^\alpha\}} u_n^\gamma v_n^\delta \, d\mu_n < 0$$

2. If $\alpha \geqslant \varphi'(\gamma+0)$ and $\delta > f_\gamma(\alpha)$, then

$$\limsup \frac{1}{\lambda_n} \log \int_{X_n \cap \{u_n \leqslant v_n^\alpha\}} u_n^\gamma v_n^\delta \, d\mu_n < 0$$

Proof. First, suppose we have $\alpha \leqslant \varphi'(\gamma-0)$ and $\delta > f_\gamma(\alpha)$. Choose $t > 0$ such that $C(\gamma+t, -\delta+\alpha t) < 0$. Then

$$\begin{aligned}\int_{X_n \cap \{u_n \geqslant v_n^\alpha\}} u_n^\gamma v_n^\delta \, d\mu_n &= \int_{X_n \cap \{u_n \geqslant v_n^\alpha\}} u_n^\gamma v_n^{\alpha t} v_n^{\delta-\alpha t} \, d\mu_n \\ &\leqslant \int_{X_n} u_n^{\gamma+t} v_n^{\delta-\alpha t} \, d\mu_n \\ &= \exp \lambda_n C_n(\gamma+t, -\delta+\alpha t)\end{aligned}$$

which proves assertion 1.

Suppose now we have $\alpha \geqslant \varphi'(\gamma+0)$ and $\delta > f_\gamma(\alpha)$. Choose $t > 0$ such that $C(\gamma-t, -\delta-\alpha t) < 0$. Then

$$\begin{aligned}\int_{X_n \cap \{u_n \leqslant v_n^\alpha\}} u_n^\gamma v_n^\delta \, d\mu_n &= \int_{X_n \cap \{u_n \leqslant v_n^\alpha\}} u_n^\gamma v_n^{-\alpha t} v_n^{\delta+\alpha t} \, d\mu_n \\ &\leqslant \int_{X_n} u_n^{\gamma-t} v_n^{\delta+\alpha t} \, d\mu_n \\ &= \exp \lambda_n C_n(\gamma-t, -\delta-\alpha t)\end{aligned}$$

which proves assertion 2.

We can also remark that, if $\delta > -\varphi(\gamma)$, we have

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \log \int_{X_n} u_n^\gamma v_n^\delta \, d\mu_n < 0$$

Proposition 2. For any $\gamma \in I$, we have the following facts:

1. If $\alpha < \varphi'(\gamma+0)$, then

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \log \int_{X_n \cap \{u_n \geqslant v_n^\alpha\}} u_n^\gamma v_n^{-\varphi(\gamma)} \, d\mu_n < 0$$

2. If $\alpha > \varphi'(\gamma-0)$, then

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \log \int_{X_n \cap \{u_n \leqslant v_n^\alpha\}} u_n^\gamma v_n^{-\varphi(\gamma)} \, d\mu_n < 0$$

Proof. Under the hypothesis of assertions 1 and 2, we have $-\varphi(\gamma) > f_\gamma(\alpha)$. So, we can use Proposition 1.

Corollary. Suppose we have $\mu = \mu_n$ for every n, $\varphi(0) = 0$, and $\sum_{n \geqslant 1} \exp -\eta\lambda_n < \infty$ for any $\eta > 0$. We then have

$$\varphi'(0+) \leqslant \liminf_{n \to \infty} \frac{\log u_n}{\log v_n} \leqslant \limsup_{n \to \infty} \frac{\log u_n}{\log v_n} \leqslant \varphi'(0-)$$

μ-almost everywhere on the set $\liminf_{n \to \infty} X_n$.

Proof. By Proposition 2, if $\alpha < \varphi'(0+)$, we have

$$\sum_{n \geqslant 1} \mu\left(X_n \cap \left\{\frac{\log u_n}{\log v_n} \leqslant \alpha\right\}\right) < \infty$$

So,

$$\liminf_{n \to \infty} \frac{\log u_n}{\log v_n} \geqslant \alpha, \qquad \mu\text{-almost everywhere on} \quad \liminf_{n \to +\infty} X_n$$

Similarly, if $\alpha > \varphi'(0^-)$, then

$$\limsup_{n \to +\infty} \frac{\log u_n}{\log v_n} \leqslant \alpha, \qquad \mu\text{-almost everywhere on} \quad \liminf_{n \to +\infty} X_n$$

3. UPPER BOUNDS FOR DIMENSIONS

Let $\{\{I_{n,j}\}_{1 \leqslant j \leqslant v_n}\}_{n \geqslant 1}$ be a sequence of partitions of the interval $[0, 1[$, each $I_{n,j}$ being an interval, semiopen to the right. In the present section, these partitions need not be nested. If $t \in [0, 1[$, $I_n(t)$ stands for that interval among $\{I_{n,j}\}_{1 \leqslant j \leqslant v_n}$ that contains t. The length of an interval J is denoted by $|J|$. We assume that, for any $t \in [0, 1[$, we have $\lim_{n \to \infty} |I_n(t)| = 0$.

We consider two dimensional indices dim and Dim which are defined in a similar way to Hausdorff and Tricot dimensions, but by considering only coverings or packings by intervals $\{I_{n,j}\}_{n \geqslant 1, 1 \leqslant j \leqslant v_n}$. The Hausdorff dimension is well known. The Tricot one is less known, so we give a survey of its definition and properties in the Appendix. We think that the Tricot dimension is of great interest. Indeed, this dimension is one of those that gives a mathematical meaning to assertions of some physicists on the "box dimension" of certain sets which are dense in an open set of some $\mathbb{R}^d$.

Let μ be a Borel probability measure on $[0, 1[$. For $\alpha \in \mathbb{R}$, let us consider the following sets:

$$B_\alpha = \left\{ t \in [0, 1[; \limsup \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \leqslant \alpha \right\}$$

$$B_\alpha^* = \left\{ t \in [0, 1[; \liminf \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \leqslant \alpha \right\}$$

$$V_\alpha = \left\{ t \in \operatorname{supp} \mu; \liminf \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \geqslant \alpha \right\}$$

$$V_\alpha^* = \left\{ t \in \operatorname{supp} \mu; \limsup \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \geqslant \alpha \right\}$$

and

$$E_{\alpha,\beta} = V_\alpha \cap B_\beta \qquad (\text{for } \alpha \leqslant \beta)$$

We are given a sequence $\{\lambda_n\}_{n \geqslant 1}$ of positive numbers such that $\sum_{n \geqslant 1} \exp -\eta\lambda_n < \infty$ for any $\eta > 0$.

We consider the following quantities:

$$C_n(x, y) = \lambda_n^{-1} \log \sum_{1 \leqslant j \leqslant v_n}' \mu(I_{n,j})^{x+1} |I_{n,j}|^{-y}$$

and

$$C(x, y) = \limsup_{n \to \infty} C_n(x, y)$$

where $\sum'$ means that the summation runs over the j's such that $\mu(I_{n,j}) \neq 0$.

These quantities are the same as those introduced in Section 2: take $\mu_n = \mu$ (for every $n \geqslant 1$), $u_n(t) = \mu(I_n(t))$, and $v_n(t) = |I_n(t)|$. As previously, we consider the function φ and the various objects attached to it. We suppose that φ is finite on an open interval containing 0 and 1. But, instead of writing f_{-1}, we shall simply write f.

When all the intervals $\{I_{n,j}\}_{1 \leqslant j \leqslant v_n}$ have the same length $\exp -\lambda_n$, we have the following relation between our function φ and the function τ described in Section 1: $\varphi(x) = \tau(x+1)$.

The following theorem provides upper bounds for the Hausdorff and Tricot dimensions of the sets B_α, B_α^*, V_α, and V_α^*.

Theorem 1.

1. For any α, we have $\operatorname{Dim} B_\alpha^* \leqslant -\varphi(-1)$ and $\operatorname{Dim} V_\alpha^* \leqslant -\varphi(-1)$.
2. If $\alpha \leqslant \varphi'(-1-0)$, then $\operatorname{Dim} B_\alpha \leqslant f(\alpha)$ and $\dim B_\alpha^* \leqslant f(\alpha)$.
3. If $\alpha \geqslant \varphi'(-1+0)$, then $\operatorname{Dim} V_\alpha \leqslant f(\alpha)$ and $\dim V_\alpha^* \leqslant f(\alpha)$.

Proof. *Assertion 1.* If $\delta > -\varphi(-1)$, it results from the remark following Proposition 1, Section 2, that $\sum_{n \geqslant 1} \sum_j' |I_{n,j}|^{\delta} < \infty$. Let X denote the set $\liminf_{n \to \infty} X_n$ (see Section 2). This set differs from the support of μ only by a countable set, and contains B_α^*. If $\{I_j\}_{j \geqslant 1}$ is any packing of X by intervals in the family $\{I_{n,j}\}_{n,j}$, we have $\sum |I_j|^{\delta} < \infty$, so $\operatorname{Dim} X < \delta$. We therefore have $\operatorname{Dim} X \leqslant -\varphi(-1)$.

Assertion 2. It is enough to consider the case $\alpha < \varphi'(-1-0)$. Set

$$B_\beta(n) = \{t \in [0, 1[;\ \mu(I_n(t)) \geqslant |I_n(t)|^{\beta}\}$$

We have

$$B_\alpha = \bigcap_{\alpha < \beta < \varphi'(-1-0)} \bigcup_{m \geqslant 1} \bigcap_{n \geqslant m} B_\beta(n)$$

and

$$B_\alpha^* = \bigcap_{\alpha < \beta < \varphi'(-1-0)} \bigcap_{m \geqslant 1} \bigcup_{n \geqslant m} B_\beta(n)$$

Let us fix $\beta \in]\alpha, \varphi'(-1-0)[$ and consider the family $\mathscr{I}$ of those $I_{n,j}$ such that $\mu(I_{n,j}) \geqslant |I_{n,j}|^{\beta}$. By Proposition 1, Section 2, if $\delta > f(\beta)$, we have

$$\sum_{I \in \mathscr{I}} |I|^{\delta} < \infty$$

Since any packing $\{I_j\}_j$ of $\bigcap_{n \geqslant m} B_\beta(n)$ such that $|I_j| \leqslant \min\{|I_{n,k}|;\ n < m,\ 1 \leqslant k < \nu_n\}$ is extracted from $\mathscr{I}$, we have $\operatorname{Dim} \bigcap_{n \geqslant m} B_\beta(n) \leqslant \delta$ for any m and $\delta > f(\beta)$. Therefore

$$\operatorname{Dim}\left(\bigcup_{m} \bigcap_{n \geqslant m} B_\beta(m)\right) \leqslant f(\beta)$$

and

$$\operatorname{Dim} B_\alpha \leqslant \inf_{\alpha < \beta < \varphi'(-1-0)} f(\beta) = f(\alpha)$$

On the other hand, the family $\{I \in \mathscr{I};\ |I| < \varepsilon\}$ covers $\bigcap_{m \geqslant 1} \bigcup_{n \geqslant m} B_\beta(n)$ for any $\varepsilon > 0$. Therefore,

$$\dim \bigcap_{m \geqslant 1} \bigcup_{n \geqslant m} B_\beta(n) \leqslant f(\beta)$$

and

$$\dim B_\alpha^* \leqslant \inf_{\alpha < \beta < \varphi'(-1-0)} f(\beta) = f(\alpha)$$

Assertion 3. It is enough to consider the case $\alpha > \varphi'(-1-0)$. Set

$$V_\beta(n) = \{t \in X;\ \mu(I_n(t)) \leqslant |I_n(t)|^\beta\}$$

We have

$$V_\alpha = \bigcap_{\varphi'(-1-0)<\beta<\alpha} \ \bigcup_{m \geqslant 1} \ \bigcap_{n \geqslant m} V_\beta(n)$$

$$V_\alpha^* = \bigcap_{\varphi'(-1-0)<\beta<\alpha} \ \bigcap_{m \geqslant 1} \ \bigcup_{n \geqslant m} V_\beta(n)$$

Let us fix $\beta \in]\varphi'(-1-0), \alpha[$ and consider the family $\mathscr{I}$ of those $I_{n,j}$ such that $0 < \mu(I_{n,j}) \leqslant |I_{n,j}|^\beta$. By Proposition 1, Section 2, if $\delta > f(\beta)$, we have $\sum_{I \in \mathscr{I}} |I|^\delta < \infty$. And we conclude as for assertion 2.

4. LOWER BOUNDS FOR DIMENSIONS

In this section, we furthermore assume that the set of intervals $\{I_{n,j}\}_{n,j}$ considered in Section 3 is endowed with the structure of a homogeneous tree: any $I_{n,j}$ contains q intervals $I_{n+1,k}$, and any $I_{n+1,k}$ is contained in one $I_{n,j}$. In these conditions, we can label the $I_{n,j}$, for $1 \leqslant j \leqslant q^n$, in the following way: $I_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n}$, with $0 \leqslant \varepsilon_j < q$.

We introduce the following notation: for nonnegative functions u and v, $u \approx v$ means that there exists a positive constant A such that $A^{-1}u \leqslant v \leqslant Au$.

We still are given a probability measure μ on the Borel sets in $[0, 1[$, and throughout this section we make the following assumptions:

H_1 $\quad \mu(I_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m, \eta_1, \ldots, \eta_n}) \approx \mu(I_{\varepsilon_1, \ldots, \varepsilon_m})\, \mu(I_{\eta_1, \ldots, \eta_n})$

H_2 $\quad |I_{\varepsilon_1, \ldots, \varepsilon_m, \eta_1, \ldots, \eta_n}| \approx |I_{\varepsilon_1, \ldots, \varepsilon_m}|\, |I_{\eta_1, \ldots, \eta_n}|$

H_3 $\quad \limsup \dfrac{1}{n} \log \left(\sup_{1 \leqslant j \leqslant q^n} |I_{n,j}| \right) < 0$

(or course the constants A_1 and A_2, implicit in H_1 and H_2, are independent of the indices involved).

In this section, we take $\lambda_n = n$. Then a subaddivity argument shows that hypotheses H_1 and H_2 imply that

$$C_n(x, y) = \frac{1}{n} \log \sum_j{}' \mu(I_{n,j})^{x+1}\, |I_{n,j}|^{-y}$$

has a finite limit $C(x, y)$ for any $(x, y) \in \mathbb{R}^2$.

We have $C(0, 0)=0$ and $C(-1, -1)\leqslant 0$. So, the corresponding φ does not assume the value $+\infty$. Moreover, it results from H_3 that Ω is not empty.

On the other hand, Michon[19] proved that, for any $(x_0, y_0)\in\mathbb{R}^2$, there exists a probability measure μ_{x_0, y_0} on $[0, 1[$ with the following property:

$$\mu_{x_0, y_0}(I_{n,j})\approx\mu(I_{n,j})^{x_0+1}\,|I_{n,j}|^{-y_0}e^{-nC(x_0, y_0)}$$

This measure, called the Gibbs measure, also satisfies H_1. So, we can consider the following quantity:

$$C_{x_0, y_0}(x, y)=\lim_{n\to\infty}\frac{1}{n}\log{\sum_j}'\mu(I_{n,j})^x\,|I_{n,j}|^{-y}\,\mu_{x_0, y_0}(I_{n,j})$$

Indeed we have

$$C_{x_0, y_0}(x, y)=\lim_{n\to\infty}\frac{1}{n}\log\int_{\{\mu(I_n(t))>0\}}\mu(I_n(t))^x\,|I_n(t)|^{-y}\,d\mu_{x_0, y_0}(t)$$

It results from a straightforward computation that

$$C_{x_0, y_0}(x, y)=C(x+x_0, y+y_0)-C(x_0, y_0)$$

Theorem 2. For any real number θ, we have

$$\dim(V_{\varphi'(\theta+0)}\cap B_{\varphi'(\theta-0)})\geqslant\begin{cases}f(\varphi'(\theta+0)) & \text{if }\ \theta\geqslant -1\\ f(\varphi'(\theta-0)) & \text{if }\ \theta\leqslant -1\end{cases}$$

Proof. Set $\mu_\theta=\mu_{\theta,\varphi(\theta)}$ and $C_\theta=C_{\theta,\varphi(\theta)}$. We then have $C_\theta(x, y)=C(x+\theta,\ y+\varphi(\theta))$. So, the function φ_θ the graph of which separates positive and negative values of C_θ is the function $\varphi_\theta(x)=\varphi(x+\theta)-\varphi(\theta)$. It results from the corollary to Proposition 2, Section 2, that, for μ_θ-almost every t, we have

$$\varphi'(\theta+0)\leqslant\liminf\frac{\log\mu(I_n(t))}{\log|I_n(t)|}\leqslant\limsup\frac{\log\mu(I_n(t))}{\log|I_n(t)|}\leqslant\varphi'(\theta-0)$$

In other terms, this means

$$\mu_\theta(V_{\varphi'(\theta+0)}\cap B_{\varphi'(\theta-0)})=1$$

On the other hand, we have

$$\frac{\log\mu_\theta(I_n(t))}{\log|I_n(t)|}=(\theta+1)\frac{\log\mu(I_n(t))}{\log|I_n(t)|}-\varphi(\theta)+o(1)$$

So, for $\theta \geqslant -1$, and for μ_θ-almost every t, we have

$$\liminf \frac{\log \mu_\theta(I_n(t))}{\log |I_n(t)|} \geqslant (\theta+1)\,\varphi'(\theta+0) - \varphi(\theta) = f(\varphi'(\theta+0))$$

Similarly, if $\theta \leqslant -1$, then, for μ_θ-almost every t, we have

$$\liminf \frac{\log \mu_\theta(I_n(t))}{\log |I_n(t)|} \geqslant (\theta+1)\,\varphi'(\theta-0) - \varphi(\theta) = f(\varphi'(\theta-0))$$

We conclude by using the Kinney–Pitcher–Billingsley theorem.[4]

Remark. In fact, we proved a bit more: if A is a Borel set such that $\mu_\theta(A) > 0$, then we have $\dim A \geqslant \min(f(\varphi'(\theta-0)), f(\varphi'(\theta+0)))$.

As a matter of fact, the above analysis has the following by-product: the existence of μ_θ for all θ's implies that f cannot assume negative values. This means that φ is defined on the whole of $\mathbb{R}$ and that its graph has two asymptotes.

In order to summarize these results, it is convenient to introduce the following notations: $\alpha_0^+ = \varphi'(-1-0)$, $\alpha_0^- = \varphi'(-1+0)$, and, if $\alpha \in [\varphi'(\theta+0), \varphi'(\theta-0)]$, we set $\alpha^+ = \varphi'(\theta-0)$ and $\alpha^- = \varphi'(\theta+0)$. By putting together lower and upper bounds, we obtain the follows results.

Theorem 3.

1. $\inf[f(\alpha^-), f(\alpha^+)] \leqslant \dim E_{\alpha^-,\alpha^+} \leqslant \operatorname{Dim} E_{\alpha^-,\alpha^+} \leqslant \sup[f(\alpha^-), f(\alpha^+)]$.
2. If $\alpha \leqslant \alpha_0^+$, then we have
$$\dim V_\alpha = \operatorname{Dim} V_\alpha^* = -\varphi(-1)$$
$$\dim B_{\alpha^-} = \operatorname{Dim} B_{\alpha^-} = \dim B_{\alpha^-}^* = f(\alpha^-)$$
3. If $\alpha \geqslant \alpha_0^-$, then we have
$$\dim B_\alpha = \operatorname{Dim} B_\alpha^* = -\varphi(-1)$$
$$\dim V_{\alpha^+} = \operatorname{Dim} V_{\alpha^+} = \dim V_{\alpha^+}^* = f(\alpha^+)$$

Proof. Assertion 1 follows easily by combining Theorems 3.1 and 4.1. Let us consider the case $\alpha \leqslant \alpha_0^+$. We have, by Theorem 3.1, $\dim B_{\alpha^-}^* \leqslant f(\alpha^-)$ and $\operatorname{Dim} B_{\alpha^-} \leqslant f(\alpha^-)$. On the other hand, if $\beta < \alpha^-$, we have $B_{\alpha^-}^* \supset B_{\alpha^-} \supset E_{\beta^-,\beta^+}$, so $\dim E_{\alpha^-} \geqslant f(\beta^-)$. But $\lim_{\beta \nearrow \alpha^-} \beta^- = \alpha^-$; therefore

$$\dim B_{\alpha^-}^* \geqslant \dim B_{\alpha^-} \geqslant \sup_{\beta < \alpha^-} f(\beta^-) = f(\alpha^-)$$

This proves the second part of assertion 2.

Now, if $\alpha \geqslant \alpha_0^-$, then we have $B_\alpha^* \supset B_\alpha \supset B_\beta$ for all $\beta \leqslant \alpha_0^-$, so

$$\dim B_\alpha \geqslant \sup_{\beta < \alpha_0^-} \dim B_{\beta^-} = \sup_{\beta < \alpha_0^-} f(\beta^-) = f(\alpha_0^-) = -\varphi(-1)$$

This proves the first part of assertion 3. The other assertions are proved similarly.

Corollary 1. Set

$$E_\alpha = \left\{ t \in [0, 1[;\ \lim_{n\to\infty} \frac{\log \mu(I_n(t))}{\log |I_n(t)|} = \alpha \right\}$$

We then have $\dim E_{\alpha^-} = \mathrm{Dim}\ E_{\alpha^-} = f(\alpha^-)$ or $\dim E_{\alpha^+} = \mathrm{Dim}\ E_{\alpha^+} = f(\alpha^+)$ according to whether $\alpha \leqslant \alpha_0^-$ or $\alpha \geqslant \alpha_0^+$.

Corollary 2. Set

$$\tilde{B}_\alpha = \left\{ t \in [0, 1];\ \limsup_{n\to\infty} \frac{\log \mu(I_n(t))}{\log |I_n(t)|} = \alpha \right\}$$

We have $\dim \tilde{B}_{\alpha^-} = \mathrm{Dim}\ \tilde{B}_{\alpha^-} = f(\alpha^-)$ or $\dim \tilde{B}_{a^+} = f(\alpha^+)$ according to whether $\alpha \leqslant \alpha_0^-$ or $\alpha \geqslant \alpha_0^+$.

This results from Theorem 2 and from the fact that $\tilde{B}_\alpha = B_\alpha \cap V_\alpha^*$. We also have a result of the same kind for the set $\tilde{V}_\alpha$ of t's for which the lower limit is α.

This last corollary generalizes a result of Collet *et al.*[7]

5. EXAMPLES

1. An Example Where $\dim E_a \neq f(a)$ for Some a. Let $\{I_{\varepsilon_1,\dots,\varepsilon_n}\}_{n \geqslant 1, 0 \leqslant \varepsilon_j \leqslant 4}$ be the collection of 5-adic intervals:

$$I_{\varepsilon_1,\dots,\varepsilon_n} = \left[\sum_{j=1}^{n} \varepsilon_j 5^{-j}, \sum_{j=1}^{n} \varepsilon_j 5^{-j} + 5^{-n} \right[$$

For any number $t \in [0, 1[$, we consider its base-5 expansion

$$t = \sum_{j \geqslant 1} \varepsilon_j 5^{-j}, \qquad 0 \leqslant \varepsilon_j < 5$$

(multiple expansions are too scarce to matter), and set

$$\varphi_j(t, n) = \frac{1}{n} \operatorname{card}\{k \leqslant n;\ \varepsilon_k = j\} \qquad \text{for} \quad 0 < j < 5$$

We define a measure μ in the following way. It is the average of two probabilities, one of which is supported by the first fifth of the unit interval and the second by the last fifth. The first of these mass distributions sees only the digits 0, 2, 4 of the base-5 expansion and treats them equally; the second sees only the digits 1, 3 and treats them equally. So, the measure μ is supported by the union of two Cantor-like sets, and for t's in its support we have

$$\mu(I_n(t)) = \begin{cases} 0 & \text{if } 1 \leqslant \varepsilon_1 \leqslant 3 \\ 2^{-1} 3^{-n(\varphi_0(t,n)+\varphi_2(t,n)+\varphi_4(t,n))} & \text{if } \varepsilon_1 = 0 \\ 2^{-1-n(\varphi_1(t,n)+\varphi_3(t,n))} & \text{if } \varepsilon_1 = 4 \end{cases}$$

It is easy to see that

$$\varphi(x) = \min\left(x \frac{\log 2}{\log 5}, x \frac{\log 3}{\log 5}\right)$$

so one has $f(\alpha) = \alpha$ for $\log 2/\log 5 \leqslant \alpha \leqslant \log 3/\log 5$. But $E_\alpha = \varnothing$ unless $\alpha = \log 2/\log 5$ or $\alpha = \log 3/\log 5$.

2. The Multinomial Measures. Let b be an integer $\geqslant 2$ and $\{I_{\varepsilon_1,\ldots,\varepsilon_n}\}$ stand for the b-adic intervals: $0 \leqslant \varepsilon_j < b$, $|I_{\varepsilon_1,\ldots,\varepsilon_n}| = b^{-n}$. We define $\varphi_j(t, n)$ ($t \in [0, 1[$, $0 \leqslant j < b$) as in the preceding paragraph.

Let $m = \{m_j\}_{0 \leqslant j < b}$ be a sequence of b nonnegative real numbers such that

$$\sum_{0 \leqslant j < b} m_j = 1$$

Then, we define a measure μ_m in the following way:

$$\log \mu_m(I_n(t)) = n \sum_{0 \leqslant j < b} \varphi_j(t, n) \log m_j$$

These measures have been used as a paradigm for multifractal measures.[12] For them, it has been proved[12] that the multifractal formalism works. As they satisfy hypothesis H_1 of Section 4, they can be handled by our method. In fact all computations are explicit: $\tau(q) = -\log(\sum'_{0 \leqslant j < b} m_j^q)$, where log is the base b logarithm. The Gibbs measures are also multinomial measures.

In the case $b = 2$, the sets E_α, $\tilde{B}_\alpha$, and V_α^* have been considered by Eggleston,[8] Besicovitch,[3] and Volkman,[24] respectively, and they determined their dimensions. So our results can be considered as a generalization of theirs, although the methods are different.

3. Besicovitch- and Volkman-Type Results for Base3. We have already defined $\varphi_0(t, n)$, $\varphi_1(t, n)$, and $\varphi_3(t, n)$. Let us consider the set

$$G_{\alpha,\beta} = \{t \in [0, 1[; \limsup_{n\to\infty} \varphi_0(t, n) \leqslant \alpha \text{ and } \limsup_{n\to\infty} \varphi_1(t, n) \leqslant \beta\}$$

Let us set

$$h(u, v, w) = -u \log u - v \log v - w \log w \qquad \text{(base 3 logarithms)}$$

We are going to prove that

$$\dim G_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha \geqslant \frac{1}{3} \text{ and } \beta \geqslant \frac{1}{3} \\ h(\alpha, \beta, 1-\alpha-\beta) & \text{if } 2\alpha+\beta \leqslant 1 \text{ and } \alpha + 2\beta \leqslant 1 \\ h\left(\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2}\right) & \text{if } \alpha + 2\beta \geqslant 1 \text{ and } \alpha \leqslant \frac{1}{3} \\ h\left(\frac{1-\beta}{2}, \beta, \frac{1-\beta}{2}\right) & \text{if } 2\alpha + \beta \geqslant 1 \text{ and } \beta \leqslant \frac{1}{3} \end{cases}$$

Clearly, if α and β are greater than 1/3, $G_{\alpha,\beta}$ contains the numbers which are normal in base 3. So, $\dim G_{\alpha,\beta} = 1$.

Let us suppose that $2\alpha + \beta \leqslant 1$ and $\alpha + 2\beta \leqslant 1$ and consider the multinomial measure $\mu = \mu_{(\alpha,\beta,1-\alpha-\beta)}$. Set $d = h(\alpha, \beta, 1-\alpha-\beta)$.

Since, for almost every t, $\varphi_0(t, n)$ and $\varphi_1(t, n)$ converge toward α and β, respectively, we have $\mu(G_{\alpha,\beta}) = 1$, and, therefore (by the remark following Theorem 4.1), $\dim G_{\alpha,\beta} \geqslant d$.

On the other hand, we have

$$-\frac{1}{n} \log \mu(I_n(t)) = \varphi_0(t, n) \log \frac{1-\alpha-\beta}{\alpha} + \varphi_1(t, n) \log \frac{1-\alpha-\beta}{\beta} - \log(1-\alpha-\beta)$$

But both numbers $(1-\alpha-\beta)/\alpha$ and $(1-\alpha-\beta)/\beta$ are larger than 1, so, if $t \in G_{\alpha,\beta}$, we have

$$\limsup_{n\to\infty} \frac{\log \mu(I_n(t))}{\log |I_n(t)|} \leqslant \alpha \log \frac{1-\alpha-\beta}{\alpha} + \beta \log \frac{1-\alpha-\beta}{\beta} - \log(1-\alpha-\beta)$$
$$\leqslant h(\alpha, \beta, 1-\alpha-\beta)$$

So $G_{\alpha,\beta} \subset B_d$. But $f(d) = d$, so $\operatorname{Dim} G_{\alpha,\beta} \leqslant d$.

In the other cases, the proof is similar, but this time we use for μ one of the measures

$$\mu_{\alpha,(1-\alpha)/2,(1-\alpha)/2} \qquad \text{or} \qquad \mu_{(1-\beta)/2,\beta,(1-\beta)/2}$$

4. The Cookie-Cutter. The final example which we would like to use to illustrate the previous results is the "cookie-cutter" introduced and elucidated by Bohr and Rand.[5] In that case the map φ is well-behaved and gives detailed information about dim E_α as a function of α.

In fact one considers a smooth expanding map F, defined on two subintervals I_0 and I_1 of the unit interval I, so that $F(I_0)=F(I_1)=I$ and denotes by F_0 and F_1 the respective inverse branches.

Define the intervals $I_{\varepsilon_1,\ldots,\varepsilon_n}=F_{\varepsilon_n}\circ\cdots\circ F_{\varepsilon_1}(I)$ and consider the Cantorlike set $\bigcap_n \bigcup_{\varepsilon_1,\ldots,\varepsilon_n} I_{\varepsilon_1,\ldots,\varepsilon_n}$ provided with the measure of maximum entropy μ: $\mu(I_{\varepsilon_1,\ldots,\varepsilon_n})=2^{-n}$. Then

$$C_n(x, y)=\frac{1}{n}\log_2 \sum_{\varepsilon_1,\ldots,\varepsilon_n} \frac{2^{-n(x+1)}}{|I_{\varepsilon_1,\ldots,\varepsilon_n}|^y}$$

As n goes to infinity, $C_n(x, y)$ converges to $C(x, y)= -(x+1)+\log_2 \rho(L_y)$, where $\rho(L_y)$ is the spectral radius of the positive transfer operator

$$L_y(h)(s)=\sum_{0,1} \frac{h(F_i(s))}{|DF_i(s)|^y}$$

Since this operator has a simple eigenvalue at the spectral radius, it results that this spectral radius is smooth as a function of y and is in fact invertible. In this case the function φ, defined by the equality $C(x, \varphi(x))=0$, it itself smooth and the previous theorems give the value of dim E_α.

APPENDIX. THE TRICOT DIMENSION

Let E be a subset of a metric space (X, d). An ε-packing of E is a collection $\{B_n\}$ of mutually disjoint closed balls of diameter less than ε which intersect E. If α is a positive number, we consider the following quantity:

$$p_\alpha(E)=\lim_{\varepsilon\to 0}\inf\left\{\sum B_n \mid B_n \text{ being an } \varepsilon\text{-packing of } E\right\}$$

Define

$$\Delta(E)=\inf\{\alpha \mid p_\alpha(E)=0\}$$

In the case $X=\mathbb{R}$, $\Delta(E)$ is nothing but the so-called "box dimension" of E. The point, with this notion of dimension, is that it does not distinguish a set and its closure. For instance, the box dimension of the rational numbers is 1, although this set is countable.

In order to obviate this difficulty, C. Tricot set the following definition:

$$\mathrm{Dim}(E) = \inf\left\{\sup_n \Delta(E_n) \mid E \subset \bigcup E_n\right\}$$

This new index has the same stability properties as the Hausdorff dimension: $A \subset B$ implies $\mathrm{Dim}\, A \leqslant \mathrm{Dim}\, B$, and if E is the union of a countable sequence $\{E_n\}$, we have $\mathrm{Dim}\, E = \sup \mathrm{Dim}\, E_n$.

On the other hand, we always have $\dim \leqslant \Delta$ (where dim stands for the Hausdorff dimension). It results that we have $\dim \leqslant \mathrm{Dim}$.

For a complete treatment of these indices see refs. 21–23, and ref. 20 for a related notion.

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